SANDIA REPORT

SAND2005-7746
Unlimited Release
Printed December 2005

A discourse on Galilean invariance, SUPG stabilization, and the variational multiscale framework.

Guglielmo Scovazzi

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

Sandia is a multiprogram laboratory operated by Sandia Corporation,
a Lockheed Martin Company, for the United States Department of Energy's
National Nuclear Security Administration under Contract DE-AC04-94-AL85000.

Approved for public release; further dissemination unlimited.

Sandia National Laboratories

Issued by Sandia National Laboratories, operated for the United States Department of Energy by Sandia Corporation.

Printed in the United States of America. This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from
U.S. Department of Energy
Office of Scientific and Technical Information
P.O. Box 62
Oak Ridge, TN 37831

Telephone:	(865) 576-8401
Facsimile:	(865) 576-5728
E-Mail:	reports@adonis.osti.gov
Online ordering:	http://www.doe.gov/bridge

Available to the public from
U.S. Department of Commerce
National Technical Information Service
5285 Port Royal Rd
Springfield, VA 22161

Telephone:	(800) 553-6847
Facsimile:	(703) 605-6900
E-Mail:	orders@ntis.fedworld.gov
Online ordering:	http://www.ntis.gov/ordering.htm

SAND2005-7746
Unlimited Release
Printed December 2005

A discourse on Galilean invariance, SUPG stabilization, and the variational multiscale framework.

Guglielmo Scovazzi
1431 Computational Shock- and Multi-physics Department
Sandia National Laboratories
P.O. Box 5800, MS 1110
Albuquerque, NM 87185-1110, USA
gscovaz@sandia.gov

Abstract

Galilean invariance is one of the key requirements of many physical models adopted in theoretical and computational mechanics. Spurred by recent research developments in shock hydrodynamics computations [13], a detailed analysis on the principle of Galilean invariance in the context of SUPG operators is presented. It was observed in [13] that lack of Galilean invariance can yield catastrophic instabilities in Lagrangian computations. Here, the analysis develops at a more general level, and an Arbitrary Lagrangian-Eulerian (ALE) formulation is used to explain how to consistently derive Galilean invariant SUPG operators. Stabilization operators for Lagrangian and Eulerian mesh computations are obtained as limits of the stabilization operator for the underlying ALE formulation. In the case of Eulerian meshes, it is shown that most of the SUPG operators designed for compressible flow computations to date *are not consistent* with Galilean invariance. It is stressed that Galilean invariant SUPG formulations can provide consistent advantages in the context of complex engineering applications, due to the simple modifications needed for their implementation.

Acknowledgments

This research was partially funded by the DOE NNSA Advanced Scientific Computing Program and the Computer Science Research Institute at Sandia National Laboratories. The author would like to thank Dr. John Shadid, at Sandia National Laboratories, for many, very helpful remarks which contributed to significantly improve this work. Valuable discussions with Dr. Rich Lehoucq and Dr. Pavel Bochev at Sandia National Laboratories are also aknowledged.

Contents

List of Figures

Chapter 1

Introduction

The Galilean invariance principle states that the form of the equations of motion of an isolated system should be invariant when a change of observer, consisting of a translation with constant velocity $\boldsymbol{V}^G$, is applied.

In the case of numerical computations, it is advisable for the discretized equations of motion to maintain the same invariance properties of the continuum. Bubnov- and Petrov-Galerkin finite element methods are obtained by enforcing that the *projection* of the equation of motion onto a specific test function space vanishes. In other words, the *residual* of the equation of motion is orthogonal to the test function space. In this case, the Galilean principle readily translates into the requirement that the residual of the equations must remain orthogonal to the Bubnov- and Petrov-Galerkin test spaces, after a Galilean transformation is performed. It is straightforward to prove that if the continuous equations are invariant, so are the discrete equations generated by a Bubnov- or Petrov-Galerkin method. This is due to the fact that the constant velocity $\boldsymbol{V}^G$ factors out of all the integrals in the variational statement, as will be clear from the discussion in chapter 6.

SUPG and variational multiscale stabilized methods [9, 10, 6, 7, 8, 11] can be interpreted as Petrov-Galerkin methods in which the test space depends on the *local* structure of the partial differential equations simulated. A typical stabilized method is derived from the corresponding Bubnov-Galerkin method by *perturbing* the test function space on element interiors. The stability properties of SUPG-stabilized methods depend on the structure of the test function perturbation. In this case, special care has to be taken to ensure invariance of the test function perturbation, to avoid the paradox of having the stability properties of the method depending on the *observer*. As was shown in [13], "standard" stabilization procedures – which usually lack Galilean invariance – were found to generate catastrophic instabilities when applied in compressible Lagrangian hydrodynamics computations (see, e.g., Fig. 1.1).

More recent work of the author has been focusing on the development of SUPG-stabilized methods for shock hydrodynamics applications on arbitrary Lagrangian-

Eulerian meshes, and the question of invariance was posed again. An important aspect of the ongoing investigation is related to what happens when the Eulerian rather than the Lagrangian limit of the ALE equations is taken. To the best of the author's knowledge, the *large majority* of the stabilization operators developed to date in the context of Eulerian (fixed) meshes for compressible flow computations are *not* consistent with Galilean invariance.

A new approach that obviates this issue will be presented and compared to the old approach. As a point of note, the instabilities documented in [13] manifested themselves whenever the inconsistent terms became predominant in the stabilization operator, while were absent in all other conditions. Therefore, it is quite possible that a milder form of such instabilities might have been experienced by other researchers, and erroneously attributed to "weakenesses" in the design of the stabilization tensor $\boldsymbol{\tau}$, whose definition has a substantial degree of arbitrariness. This statement cannot be made more precise, and may be considered as the author's "reasonable doubt".

It will be shown that conformity with Galilean invariance can be achieved by a number of straightforward *simplifications*, which imply a conspicuous reduction in the computational cost of the stabilization operator. As will become clear from the forthcoming discussion, it is *easier* and computationally *more efficient* to develop Galilean invariant stabilized operators, which, in addition, have the potential for improved reliability in complex geometry, multi-physics applications.

Given the fact that SUPG technology for compressible flows is well established, and its reliability seems proven, the "why bother about Galilean invariance" question can be asked. There are at least two answers: On the one hand, the main principles of physics should never be overlooked, especially given the current trend of increasing the complexity of the systems simulated: the results in [13] are evidence of the traps along the way. On the other hand, one could answer a question with a question: "Why taking risks?"

The rest of the material is organized as follows: A very general discussion of the issue of Galilean invariance in the context of ALE equations and its Eulerian and Lagrangian limits is presented in chapter 2. The ALE description of the kinematics of motion is presented in chapter 3. Chapter 4 presents an example of the invariance issue in the context of a one dimensional scalar advection equation, and a brief survey of Galilean invariant SUPG methods for incompressible flows. A stabilized space-time variational formulation of the ALE compressible Euler equations is developed in chapter 5. Chapter 6 presents an analysis of the invariance properties of the residuals and their effect on the approximation to the subgrid-scale solution. In chapter 7, a Galilean consistency analysis shows that standard SUPG formulations for compressible flows yield a non-invariant test function space. A new, invariant approach is also developed, and its advantages are analyzed in detail. Conclusions are summarized in chapter 8.

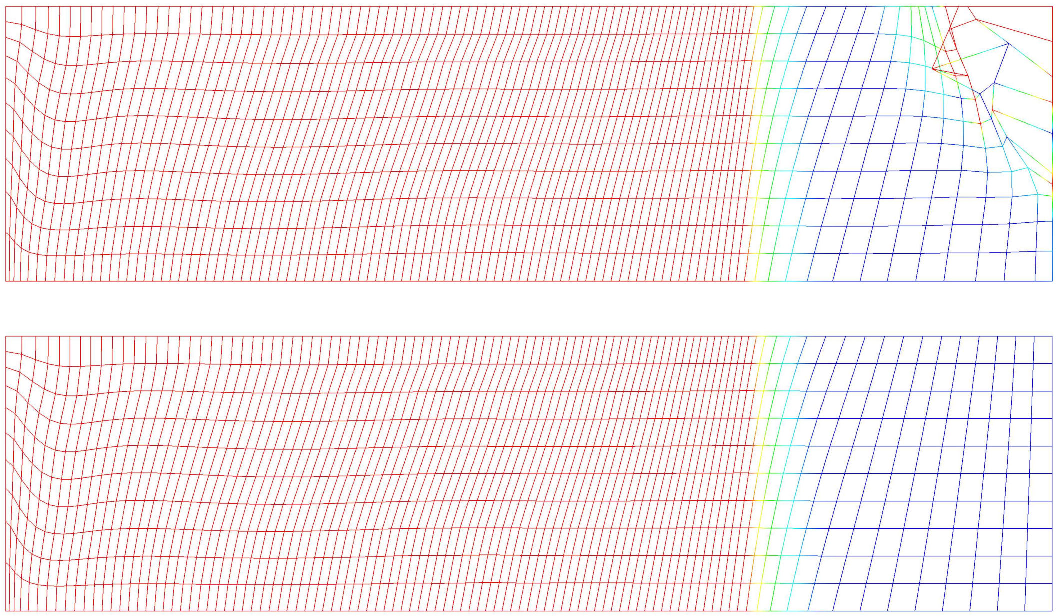

Figure 1.1. Results from the computations from [13]. Mesh distortion plot: The color scheme represents the pressure. Above: SUPG formulation violating Galilean invariance. Below: SUPG abiding the Galilean invariance principle. A classical quadrilateral Saltzmann mesh is used in an implosion computation. The initial velocity is of unit magnitude and directed horizontally from right to left, except the left boundary which is held fixed. The initial density is unity and the initial specific internal energy is 10^{-1}. A shock forms at the left boundary and advances to the right. Note the *mesh coasting* phenomenon on the top right corner of the upper domain, absent in the SUPG formulation satisfying Galilean invariance, below.

Chapter 2

Galilean transformation in the ALE context.

A simple introductory discussion on how Galilean invariance applies to SUPG formulations in various reference frames is now presented. In order to explore the computational implications of the Galilean principle, we need to think about the mesh as a (possibly moving) *laboratory* which is used to sample the numerical data. In this sense we cannot completely separate the numerical aspects from the physics of the problem, since SUPG forces act to stabilize advection and pressure perturbations *across* the mesh. Hence, when a Galilean transformation (expressed here in the Eulerian reference frame)

$$\begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} \\ -\boldsymbol{V}_G & \boldsymbol{I}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} & \boldsymbol{I}_{3\times 3} \end{bmatrix} \begin{bmatrix} t \\ \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{0}_{3\times 1} \\ \boldsymbol{V}_G \end{bmatrix} \tag{2.1}$$

$$\begin{bmatrix} t \\ \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} \\ \boldsymbol{V}_G & \boldsymbol{I}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} & \boldsymbol{I}_{3\times 3} \end{bmatrix} \begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{v}} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{0}_{3\times 1} \\ \boldsymbol{V}^G \end{bmatrix} \tag{2.2}$$

is applied to an isolated system, both the material domain and the mesh are affected. In other words, the relative velocities between the mesh and the material are preserved under the transformation.

In the generic ALE context (see Fig. 2.1), an invariant SUPG perturbation to the Bubnov-Galerkin test function *can only* depend on thermodynamic variables and their gradients, the difference $\boldsymbol{c}$ between the material velocity $\boldsymbol{v}$ and the mesh velocity $\hat{\boldsymbol{v}}$, and derivatives of the velocities and position vectors, the only invariant quantities.

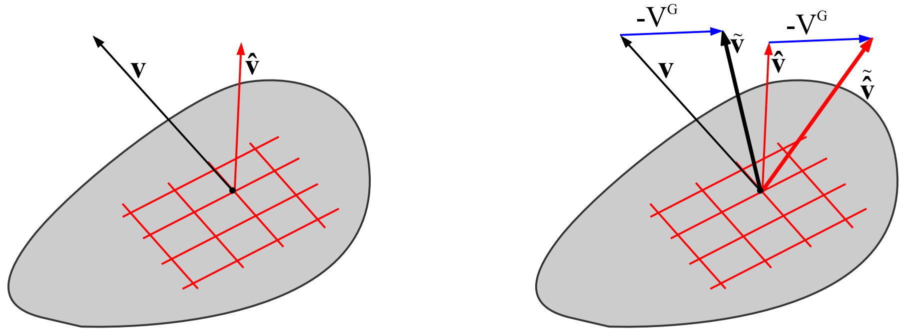

Figure 2.1. Sketch for Galilean transformation for a generic ALE mesh. Left: A material domain, and the corresponding mesh (the red grid), are moving with velocity $\boldsymbol{v}$ (black arrow) and $\hat{\boldsymbol{v}}$ (red arrow), respectively. Left: After a Galilean transformation is applied, the material and the mesh are moving with velocities $\tilde{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}^G$ and $\tilde{\hat{\boldsymbol{v}}} = \hat{\boldsymbol{v}} - \boldsymbol{V}^G$, respectively. The relative velocity of the material with respect to the mesh is invariant: $\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{v}} - \tilde{\hat{\boldsymbol{v}}} = \boldsymbol{v} - \boldsymbol{V}^G - \hat{\boldsymbol{v}} + \boldsymbol{V}^G = \boldsymbol{v} - \hat{\boldsymbol{v}} = \boldsymbol{c}$.

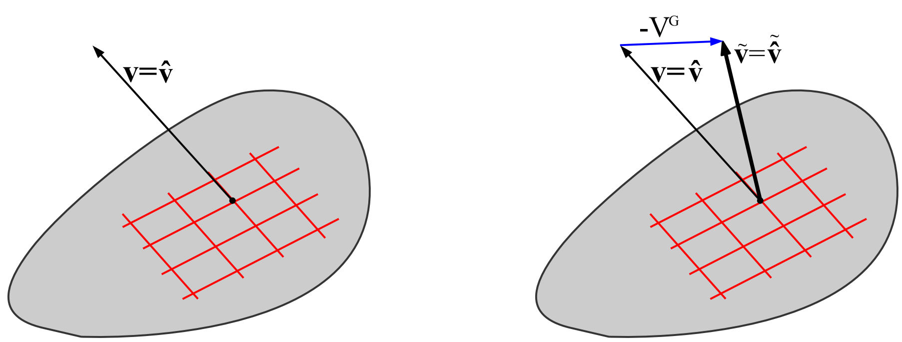

Figure 2.2. Sketch for Galilean transformation for a Lagrangian mesh. Left: A material domain, and the corresponding mesh (the red grid), are moving with the same velocity $\boldsymbol{v} = \hat{\boldsymbol{v}}$ (black arrow). Left: After a Galilean transformation is applied, the material and the mesh are moving with velocity $\tilde{\boldsymbol{v}} = \tilde{\hat{\boldsymbol{v}}} = \boldsymbol{v} - \boldsymbol{V}^G = \hat{\boldsymbol{v}} - \boldsymbol{V}^G$. Therefore a Lagrangian mesh is transformed into a Lagrangian mesh by a Galilean transformation.

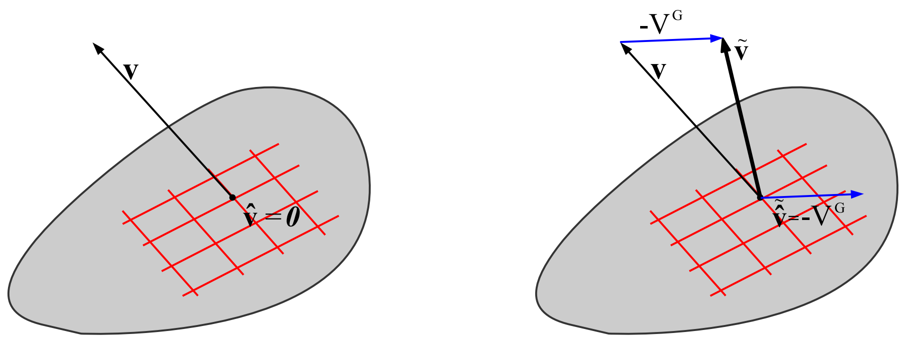

Figure 2.3. Sketch for Galilean transformation for an Eulerian mesh. Left: The material is moving with velocity $\boldsymbol{v}$, while the mesh is fixed ($\hat{\boldsymbol{v}} = 0$). Left: After a Galilean transformation is applied, the material is moving with velocity $\tilde{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}^G$ and the mesh is undergoing a rigid body translation motion with velocity $\tilde{\hat{\boldsymbol{v}}} = -\boldsymbol{V}^G$. Therefore an originally fixed mesh, after transformation, assumes constant translational motion. However, $\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{v}} - \tilde{\hat{\boldsymbol{v}}} = \boldsymbol{v} - \boldsymbol{V}^G + \boldsymbol{V}^G = \boldsymbol{v} = \boldsymbol{c}$, as for a generic ALE mesh.

In the Lagrangian limit the mesh is tied to the material ($\boldsymbol{v} = \hat{\boldsymbol{v}}$, i.e., $\boldsymbol{c} = \boldsymbol{0}$, see Fig. 2.2), and the *absolute* velocity of the material $\boldsymbol{v}$ *cannot* enter the expression for the SUPG perturbation to the test space.

In the Eulerian limit, the mesh, seen from the transformed coordinate system, is moving with constant velocity $-\boldsymbol{V}^G$ (see Fig. 2.3). Therefore, an Eulerian mesh transforms into a moving mesh after a Galilean change of coordinates is performed. This situation is not paradoxical, but a simple consequence of the general principle of invariance applied to the ALE framework.

Remark 1 *Developing SUPG operators for Eulerian meshes is somewhat problematic, since it is not possible to discern from the equations whether the meaning of "$\boldsymbol{v}$" is $\boldsymbol{v} - \hat{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{0} = \boldsymbol{c}$, a relative velocity, or simply $\boldsymbol{v}$, the absolute material velocity. In this sense the best way to develop SUPG operators for Eulerian computations is to start from the ALE formulation and then enforce a motionless mesh.*

Chapter 3

Kinematics of motion

The purpose of the present section is to fix the notation for Arbitrary Lagrangian Eulerian equations and recall a number of very important results. The notation used in [1] is adopted in what follows, with minor differences. The reader can also refer to [3] or [4] for further details. A point of departure in the discussion of the Arbitrary Lagrangian-Eulerian approach is to define the *material* (or *Lagrangian*), *referential*, and *Eulerian* reference frames. Let Ω_0, $\hat{\Omega}$, and Ω be opens in $\mathbb{R}^{n_d}$. The *deformation* $\boldsymbol{\varphi}$ is the transformation from the material to the Eulerian reference frame

$$\begin{aligned} \boldsymbol{\varphi} : \Omega_0 \quad &\rightarrow \quad \Omega = \boldsymbol{\varphi}(\Omega_0), &(3.1)\\ \boldsymbol{X} \quad &\mapsto \quad \boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t), \quad \forall \boldsymbol{X} \in \Omega_0,\ t \geq 0, &(3.2) \end{aligned}$$

Here $\boldsymbol{X}$ is the material coordinate (which usually corresponds to the point vector in the initial configuration of the body), and $\boldsymbol{x}$ is the point vector in the Eulerian frame. Ω_0 is the domain occupied by the body in the material reference frame. $\boldsymbol{\varphi}$ maps Ω_0 to Ω, the domain occupied by the body in the current configuration (Eulerian frame). It is also useful to define the *deformation gradient*, and the *Jacobian determinant*:

$$\begin{aligned} \boldsymbol{F} \quad &= \quad \nabla_{\boldsymbol{X}}\, \boldsymbol{\varphi} = \frac{\partial \varphi_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} &(3.3)\\ J \quad &= \quad \det(\boldsymbol{F}) &(3.4) \end{aligned}$$

The *referential map* $\hat{\boldsymbol{\varphi}}$, from the referential frame to the Eulerian frame, is defined as

$$\begin{aligned} \hat{\boldsymbol{\varphi}} : \hat{\Omega} \quad &\rightarrow \quad \Omega = \hat{\boldsymbol{\varphi}}(\hat{\Omega}), &(3.5)\\ \boldsymbol{\chi} \quad &\mapsto \quad \boldsymbol{x} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t), \quad \forall \boldsymbol{\chi} \in \hat{\Omega},\ t \geq 0, &(3.6) \end{aligned}$$

where $\boldsymbol{\chi}$ is the point vector in the referential frame, and $\hat{\Omega}$, the domain occupied by the body in the referential frame, is mapped to Ω by $\hat{\boldsymbol{\varphi}}$. In addition, the *mesh* *deformation gradient* and the *mesh Jacobian determinant* are defined as:

$$\hat{\boldsymbol{F}} = \nabla_{\chi} \hat{\boldsymbol{\varphi}} = \frac{\partial \hat{\varphi}_i}{\partial \chi_j} = \frac{\partial x_i}{\partial \chi_j} \tag{3.7}$$

$$\hat{J} = \det(\hat{\boldsymbol{F}}) \tag{3.8}$$

For the purpose of this paper, the referential frame of reference lies on a mesh which is not fixed in space (Eulerian) nor attached to the material (Lagrangian), but moves in time with an arbitrary motion. The transformation from the material to the referential frame will also be needed, namely

$$\boldsymbol{\psi} : \Omega_0 \rightarrow \hat{\Omega} = \boldsymbol{\psi}(\Omega_0), \tag{3.9}$$

$$\boldsymbol{X} \mapsto \boldsymbol{\chi} = \boldsymbol{\psi}(\boldsymbol{X}, t), \quad \forall \boldsymbol{X} \in \Omega_0,\ t \geq 0, \tag{3.10}$$

The definition of the *referential deformation gradient* reads

$$\nabla_{\boldsymbol{X}} \boldsymbol{\psi} = \frac{\partial \psi_i}{\partial X_j} = \frac{\partial \chi_i}{\partial X_j} \tag{3.11}$$

Displacements can then be defined as

$$\boldsymbol{u} = \boldsymbol{\varphi}(\boldsymbol{X}, t) - \boldsymbol{\varphi}(\boldsymbol{X}, 0) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X} \tag{3.12}$$

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t) - \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, 0) = \boldsymbol{x}(\boldsymbol{\chi}, t) - \boldsymbol{X} \tag{3.13}$$

with the practical assumption, $\boldsymbol{\chi}(\boldsymbol{X}, t = 0) = \boldsymbol{X}$. The referential displacement $\hat{\boldsymbol{u}}$ is the displacement undergone by the mesh. Analogously, material and mesh velocities can be defined:

$$\boldsymbol{v} = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{X}} = \left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{\boldsymbol{X}} = \dot{\boldsymbol{u}}, \tag{3.14}$$

$$\hat{\boldsymbol{v}} = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{\chi}} = \left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{\boldsymbol{\chi}} \tag{3.15}$$

Using the chain rule, it is now possible to derive two very important expressions of the Lagrangian time derivative of a scalar-valued function f:

$$\dot{f}(\boldsymbol{\chi}, t) = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{w} \cdot \nabla_{\chi} f = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{c} \cdot \nabla_{x} f \tag{3.16}$$

where ∇_{χ} and ∇_{x} are the gradients in the referential and Eulerian frames, respectively. $\boldsymbol{w} = \partial_t \boldsymbol{\chi}|_{\boldsymbol{X}} = \dot{\boldsymbol{\psi}}(\boldsymbol{X}, t) = \dot{\boldsymbol{\chi}}$ is the *particle referential velocity*, that is the velocity of a material point seen from the referential frame. The *convective velocity* $\boldsymbol{c}$ is the velocity of the material relative to the mesh, and is related to $\boldsymbol{w}$ through

$$\boldsymbol{c} \;=\; \boldsymbol{v} - \hat{\boldsymbol{v}} = \hat{\boldsymbol{F}}\boldsymbol{w} \tag{3.17}$$

or, in index notation, $c_i = v_i - \hat{v}_i = \hat{F}_{ij} w_j$, with $\hat{F}_{ij} = \partial_{\chi_j} \hat{\varphi}_i(\boldsymbol{\chi}, t) = \partial_{\chi_j} x_i$.

3.1 Limit behavior

In the Lagrangian limit, $\boldsymbol{\chi} \equiv \boldsymbol{X}$, $\hat{\boldsymbol{v}} = \boldsymbol{v}$, and $\hat{\boldsymbol{F}} \equiv \boldsymbol{F}$, $\forall t$, so that $\boldsymbol{w} = \dot{\boldsymbol{\chi}} = \dot{\boldsymbol{X}} = \boldsymbol{0}$, and, in addition, $\boldsymbol{c} = \hat{\boldsymbol{F}}\boldsymbol{w} = \boldsymbol{0}$.

In the Eulerian limit, $\boldsymbol{\chi} \equiv \boldsymbol{x}$, $\hat{\boldsymbol{v}} = \boldsymbol{0}$, and $\hat{\boldsymbol{F}} \equiv \boldsymbol{I}$, $\forall t$, so that $\boldsymbol{w} = \dot{\boldsymbol{\chi}} = \dot{\boldsymbol{x}} = \boldsymbol{v}$, and, in addition, $\boldsymbol{c} = \hat{\boldsymbol{F}}\boldsymbol{w} = \boldsymbol{I}\boldsymbol{w} = \boldsymbol{w} = \boldsymbol{v}$.

Chapter 4

Preamble: Linear scalar advection equation in one dimension

The discussion in the case of compressible flow equations will involve a large number of algebraic manipulations. However, the ideas expressed in the paper can be explained in a very simple way in the context of a scalar advection equation of the type

$$\left.\begin{array}{c} \dot{\phi} \\ \left.\dfrac{\partial \phi}{\partial t}\right|_{\chi} + w\dfrac{\partial \phi}{\partial \chi} \\ \left.\dfrac{\partial \phi}{\partial t}\right|_{x} + v\dfrac{\partial \phi}{\partial x} \end{array}\right\} = f \tag{4.1}$$

where the Lagrangian, ALE, and Eulerian descriptions of motion have been adopted. To avoid including boundary conditions in the discussion, the domains are infinite, namely, $\Omega_0 = \hat{\Omega} = \Omega = (-\infty, \infty)$. Assuming w and v constant (i.e., the material and the mesh velocities are constants, possibly different from one another) the Galerkin formulations corresponding to problem (4.1) in the three reference frames are given by:

$$0 = \int_{\Omega_0} \Psi^h \left(\dot{\phi}^h - f\right) dX \tag{4.2}$$

$$0 = \int_{\hat{\Omega}} \hat{\psi}^h \left.\frac{\partial \phi^h}{\partial t}\right|_{\chi} - \frac{\partial \hat{\psi}^h}{\partial \chi} w\phi^h - \hat{\psi}^h f \quad d\chi \tag{4.3}$$

$$0 = \int_{\Omega} \psi^h \left.\frac{\partial \phi^h}{\partial t}\right|_{x} - \frac{\partial \psi^h}{\partial x} v\phi^h - \psi^h f \quad dx \tag{4.4}$$

where ϕ^h is the numerical approximation to ϕ and Ψ^h, $\hat{\psi}^h$, and ψ^h are the test functions, collocated at the nodes of the Lagrangian, ALE, and Eulerian meshes, respectively. To fix the ideas, we can think of solving the Galerkin formulations above in the space of continuous functions which are piecewise linear over each element of the discretization. Let us then ask ourselves the question: "What is the correct way of stabilizing the Galerkin discretizations (4.2)–(4.4)?" The answer is easy in this simple case.

For the Lagrangian form (4.2), there is no advection across the mesh, and a simple ordinary differential equation does not need stabilization. Anticipating a later discussion, the case of compressible Euler equations is more complicated, since there is still no advection of material across the computational grid, but acoustic waves do propagate through the mesh and need stabilization.

For the ALE equation (4.3), the advection is given by the particle referential velocity w, and, applying the SUPG method originally developed by Brooks and Hughes in [2], the stabilization term reads:

$$\mathcal{SUPG}(\hat{\psi}^h, \phi^h) = \sum_{e=1}^{n_{el}} \int_{\hat{\Omega}^e} \left(w \frac{\partial \hat{\psi}^h}{\partial \chi} \right) \tau_e \left(\left. \frac{\partial \phi^h}{\partial t} \right|_{\chi} + w \frac{\partial \phi^h}{\partial \chi} - f \right) \, d\chi \tag{4.5}$$

where a typical choice for τ is given by

$$\tau_e = \left(\left(\frac{2}{\Delta t} \right)^{\beta} + \left| \frac{\beta w}{\Delta \chi_e} \right|^{\beta} \right)^{-1/\beta} \tag{4.6}$$

with $\beta \geq 1$. $\Delta \chi_e$ is the element length of the referential mesh.

Remark 2 *The perturbation to the test function, $w \partial_\chi \hat{\psi}^h \tau_e$, is clearly invariant. Therefore, for this simple example, the Petrov-Galerkin method generated by the SUPG approach is invariant at the discrete level. Notice that the residual $\left. \frac{\partial \phi^h}{\partial t} \right|_{\chi} + w \frac{\partial \phi^h}{\partial \chi} - f$ is also invariant. For more complicated sets of nonlinear equations, this last condition may not always be verified, as explained in chapter 6.*

In the Eulerian case, the advection is due to the material velocity v, and the derivations are analogous to the ALE case:

$$\mathcal{SUPG}(\psi^h, \phi^h) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \left(v \frac{\partial \psi^h}{\partial x} \right) \tau_e \left(\left. \frac{\partial \phi^h}{\partial t} \right|_{x} + v \frac{\partial \phi^h}{\partial x} - f \right) \, dx \tag{4.7}$$

$$\tau_e = \left(\left(\frac{2}{\Delta t} \right)^{\beta} + \left| \frac{\beta v}{\Delta x_e} \right|^{\beta} \right)^{-1/\beta} \tag{4.8}$$

Remark 3 *The Eulerian and Lagrangian cases are limits of the ALE case. In fact, (4.5) vanishes for* $w = 0$*, and transforms to (4.7), for* $w = v$*.*

The previous derivations are clearly consistent with the principle of Galilean invariance: if a Galilean transformation is applied to the Eulerian mesh, we will recover an ALE formulation as in (4.5)–(4.6), with a transformed mesh velocity $\tilde{\hat{v}} = -V^G$, so that $\tilde{w} = \tilde{v} - \tilde{\hat{v}} = v - V^G + V^G = v$. With these substitutions, the transformed SUPG operator is *exactly identical* to the original Eulerian SUPG operator, and for a very important reason: *it is the advection relative to the mesh that needs stabilization, and not the absolute advection.*

Remark 4 *It would have been utterly incorrect to say that for the transformed Eulerian case, the form of (4.7)–(4.8) would hold unchanged, with* $\tilde{v}$ *in place of* v*. In this case, the SUPG operator would change even if the advection relative to the mesh were unchanged: This is the key point of the entire discussion.*

Remark 5 *It is clear that if a general approach needs to be developed, it is crucial to start from the ALE equations and take Lagrangian and Eulerian limits, rather than trying to generalize a concept developed for the Eulerian equations.*

4.1 A brief survey on stabilized methods for incompressible flow

Although the present work is mainly focused on compressible flows, it is worthwhile to briefly discuss the incompressible case, for which *all* the most commonly used stabilization techniques *are* Galilean invariant.

4.1.1 SUPG stabilization of the incompressible Navier-Stokes equations, Brooks and Hughes [2]

Considering the incompressible Navier-Stokes equations in ALE advective form, it is straightforward to see that Galilean invariance is preserved:

$$\mathcal{SUPG}(\hat{\boldsymbol{\psi}}_{\boldsymbol{v}}^h; \rho, \boldsymbol{v}^h, \boldsymbol{w}, p^h) = \sum_{e=1}^{n_{el}} \int_{\hat{\Omega}_n^e} (\tau \boldsymbol{w} \cdot \nabla_{\boldsymbol{\chi}} \hat{\boldsymbol{\psi}}_{\boldsymbol{v}}^h) \cdot \hat{\boldsymbol{Res}}^{\boldsymbol{v}}(\rho, \boldsymbol{v}^h, \boldsymbol{w}, p^h)\, d\hat{Q} \tag{4.9}$$

Here $\hat{\boldsymbol{\psi}}_{\boldsymbol{v}}^h$ is the test function vector, and $\hat{\boldsymbol{Res}}^{\boldsymbol{v}}(\boldsymbol{v}^h, \boldsymbol{w}, p^h)$ is the Galerkin residual of the momentum equations in advective form. In the Eulerian limit, $\boldsymbol{w} = \boldsymbol{v}$, $\boldsymbol{\chi} = \boldsymbol{x}$, and the familiar expression for the stabilization is recovered. Notice that Galilean invariance holds as long as

$$\tau \;\; = \;\; \tau(\boldsymbol{w}, \nu, \Delta t, \Delta\chi_e) \tag{4.10}$$

where ν is the physical viscosity, and $\Delta\chi_e$ is a mesh length scale.

4.1.2 PSPG stabilization, Tezduyar [16]

PSPG-type terms are also Galilean consistent. In the ALE context, their form is:

$$\mathcal{PSPG}(\hat{\psi}^h_\rho; \rho, \boldsymbol{v}^h, \boldsymbol{w}, p^h) \;\; = \;\; \sum_{e=1}^{n_{el}} \int_{\hat{\Omega}^e_n} \frac{\tau_{PSPG}}{\rho} \nabla_{\boldsymbol{\chi}} \hat{\psi}^h_\rho \cdot \hat{\boldsymbol{Res}}^{\boldsymbol{v}}(\rho, \boldsymbol{v}^h, \boldsymbol{w}, p^h) \; d\hat{Q} \tag{4.11}$$

where $\hat{\psi}^h_\rho$ is the test function for the mass conservation (divergence-free velocity field constraint), and

$$\tau_{PSPG} \;\; = \;\; \tau_{PSPG}(\boldsymbol{w}, \nu, \Delta t, \Delta\chi_e) \tag{4.12}$$

4.1.3 Advanced multiscale concepts and turbulence [5]

Recent developments in the application of multiscale methods to stabilization and turbulence subgrid modeling hinge upon substituting the subgrid-scale approximation

$$\boldsymbol{v}' \;\; \approx \;\; -\tau \hat{\boldsymbol{Res}}^{\boldsymbol{v}}(\boldsymbol{v}^h, \boldsymbol{w}, p^h) \tag{4.13}$$

in the Galerkin mesh-scale equations. As long as the τ parameter is in the form (4.10), the overall approach is Galilean invariant.

Chapter 5

Arbitrary Lagrangian-Eulerian equations

The ALE equations are now derived, and discretized according to a space time formulation similar to the one developed in [14].

5.1 Generalized Reynolds transport theorem

In order to derive useful integral forms of conservation laws, a generalized version of the classical Reynolds transport theorem is needed. It is important to realize that the transport theorem is simply an integral equation between the material reference frame and an arbitrary reference frame, which may or may not correspond to the Eulerian frame. Hence, if referential coordinates are used,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\Omega}} f\hat{J}\ d\hat{\Omega} = \int_{\hat{\Omega}} \left.\frac{\partial(f\hat{J})}{\partial t}\right|_{\boldsymbol{\chi}} d\hat{\Omega} + \int_{\hat{\Gamma}=\partial\hat{\Omega}} f\boldsymbol{w}\cdot\hat{\boldsymbol{n}}\ \hat{J}\ d\hat{\Gamma} \tag{5.1}$$

Equation (5.1) can be derived [1] noticing that it corresponds to the standard form in Eulerian coordinates with $\boldsymbol{\chi}$ in place of $\boldsymbol{x}$ and $\boldsymbol{w}$ in place of $\boldsymbol{v}$.

5.2 Integral form of the ALE equations in the referential coordinate frame

Applying (5.1) to the mass, momentum, and total energy, it is easily derived (see [1], pp. 443–447):

$$0 = \int_{\hat\Omega} \left.\frac{\partial \hat\rho}{\partial t}\right|_{\boldsymbol{\chi}} d\hat\Omega + \int_{\hat\Gamma} \hat\rho \boldsymbol{w}\cdot\hat{\boldsymbol{n}}\, d\hat\Gamma \tag{5.2}$$

$$0 = \int_{\hat\Omega} \left.\frac{\partial (\hat\rho\boldsymbol{v})}{\partial t}\right|_{\boldsymbol{\chi}} d\hat\Omega + \int_{\hat\Gamma} (\hat\rho \boldsymbol{v}\otimes\boldsymbol{w} - \hat{\boldsymbol{P}})\hat{\boldsymbol{n}}\, d\hat\Gamma - \int_{\hat\Omega} \hat\rho\boldsymbol{g}\, d\hat\Omega \tag{5.3}$$

$$\begin{aligned} 0 = {} & \int_{\hat\Omega} \left.\frac{\partial (\hat\rho E)}{\partial t}\right|_{\boldsymbol{\chi}} d\hat\Omega + \int_{\hat\Gamma} (\hat\rho E\boldsymbol{w} - \hat{\boldsymbol{P}}^T\boldsymbol{v} + \hat{\boldsymbol{Q}})\cdot\hat{\boldsymbol{n}}\, d\hat\Gamma \\ & - \int_{\hat\Omega} \hat\rho(\boldsymbol{v}\cdot\boldsymbol{g} + s)\, d\hat\Omega \end{aligned} \tag{5.4}$$

where $\hat\rho = \rho\hat J$, $\boldsymbol{g}$ is the body force term per unit mass (e.g., the gravitational acceleration), $\hat{\boldsymbol{P}} = \hat J\boldsymbol{\sigma}\hat{\boldsymbol{F}}^{-T} = \boldsymbol{\sigma}\,\mathbf{cof}\hat{\boldsymbol{F}}$, $\boldsymbol{\sigma}$ is the Cauchy stress tensor in Eulerian coordinates, $E = e+\boldsymbol{v}\cdot\boldsymbol{v}/2$ is the total energy per unit mass, e is the internal energy per unit mass, $\hat{\boldsymbol{Q}} = (\boldsymbol{q}^T\mathbf{cof}\hat{\boldsymbol{F}})^T = (\mathbf{cof}\hat{\boldsymbol{F}})^T\boldsymbol{q} = \hat J\hat{\boldsymbol{F}}^{-1}\boldsymbol{q}$, $\boldsymbol{q}$ is the heat flux in the Eulerian frame, and s is a heat source ($s > 0$) or sink ($s < 0$) per unit mass. Using the following definitions

$$\hat{\boldsymbol{U}} = \begin{bmatrix} \hat J\rho \\ \hat J\rho v_1 \\ \hat J\rho v_2 \\ \hat J\rho v_3 \\ \hat J\rho E \end{bmatrix}, \qquad \boldsymbol{Y} = \begin{bmatrix} \rho \\ v_1 \\ v_2 \\ v_3 \\ p \end{bmatrix}, \qquad \hat{\boldsymbol{Z}} = \begin{bmatrix} 0 \\ -\hat J\rho g_1 \\ -\hat J\rho g_2 \\ -\hat J\rho g_3 \\ -\hat J\rho v_i g_i - \hat J\rho s \end{bmatrix} \tag{5.5}$$

$$\hat{\boldsymbol{G}}_i = \begin{bmatrix} \hat J\rho w_i \\ \hat J\rho v_1 w_i - \hat P_{1i} \\ \hat J\rho v_2 w_i - \hat P_{2i} \\ \hat J\rho v_3 w_i - \hat P_{3i} \\ \hat J\rho E w_i - v_k\hat P_{ik} + \hat Q_i \end{bmatrix} = \begin{bmatrix} \hat J\rho w_i \\ \hat J\rho v_1 w_i - \sigma_{1k}\,\mathrm{cof}\hat F_{ki} \\ \hat J\rho v_2 w_i - \sigma_{2k}\,\mathrm{cof}\hat F_{ki} \\ \hat J\rho v_3 w_i - \sigma_{3k}\,\mathrm{cof}\hat F_{ki} \\ \hat J\rho E w_i + (v_j\sigma_{jk} + q_k)\,\mathrm{cof}\hat F_{ki} \end{bmatrix} \tag{5.6}$$

with $i = 1, 2, 3$. Equations (5.2)–(5.4) can be expressed more succinctly in vector form:

$$\partial_t|_{\boldsymbol{\chi}}\hat{\boldsymbol{U}}(\boldsymbol{Y}) + \partial_{\chi_i}\hat{\boldsymbol{G}}_i(\boldsymbol{Y}) + \hat{\boldsymbol{Z}} = \mathbf{0} \tag{5.7}$$

where the Gauss divergence theorem has been applied, as well as the fact that (5.2)–(5.4) hold on an arbitrary domain. $\boldsymbol{Y}$ is the vector of solution variables, $\hat{\boldsymbol{U}}$ is the vector of conserved variables, $\hat{\boldsymbol{G}}_i$ is the Euler flux in the i-th direction, and $\hat{\boldsymbol{Z}}$ is a vector-valued source term.

5.3 Mie-Grüneisen constitutive laws

It is assumed that the materials under consideration do not possess deformation strength, so that the Cauchy stress tensor $\boldsymbol{\sigma}$ reduces to an isotropic tensor, dependent only on the thermodynamic pressure:

$$\sigma_{ij} \;=\; -p\;\delta_{ij} \tag{5.8}$$

with δ_{ij}, the Kronecker tensor. Mie-Grüneisen materials satisfy an equation of state of the form $p = f_1(\rho; \rho_r, e_r) + f_2(\rho; \rho_r, e_r)e$, where ρ_r and e_r are fixed reference thermodynamic states. More succinctly,

$$p \;=\; f_1(\rho) \;+\; f_2(\rho)\; e \tag{5.9}$$

If $f_1 = 0$ and $f_2 = (\gamma - 1)\;\rho$, the equation of state for an ideal gas, $p \;=\; (\gamma - 1)\;\rho\;e$, is recovered. Thanks to the Mie-Grüneisen constitutive equations, a quasi-linear form of (5.7) can be developed, namely,

$$\hat{\boldsymbol{A}}_0\;\partial_t|_{\boldsymbol{\chi}}\boldsymbol{Y} + \hat{\boldsymbol{A}}_i(\boldsymbol{Y})\;\partial_{\chi_i}\boldsymbol{Y} + \hat{\boldsymbol{C}}(\boldsymbol{Y})\;\boldsymbol{Y} \;\;=\;\; \boldsymbol{0} \tag{5.10}$$

The definitions of $\hat{\boldsymbol{A}}_0$, $\hat{\boldsymbol{A}}_i$, and $\hat{\boldsymbol{C}}$ will be given in chapter 7, and depend on the choice of the solution vector $\boldsymbol{Y}$.

5.4 A space-time variational formulation in referential coordinates

In order to lay the foundations for the subsequent discussion, a space-time variational formulation in the referential frame is presented. The analysis of Galilean invariance is not strictly dependent on the variational formulation adopted, and, for example, similar conclusions hold for alternative space-time or semi-discrete formulations. In this paper, the approach developed in [14] for the purely Lagrangian case is extended to the ALE equations.

Given a partition $0 < t_1 < t_2 < \ldots < t_N = T$ of the time interval $I =]0, T]$, let $I_n =]t_n, t_{n+1}]$, so that $]0, T] = \bigcup_{n=0}^{N-1} I_n$. The space-time domain $\hat{Q} = \hat{\Omega} \times I$ can be divided into time slabs

$$\hat{Q}_n = \hat{\Omega} \times I_n \tag{5.11}$$

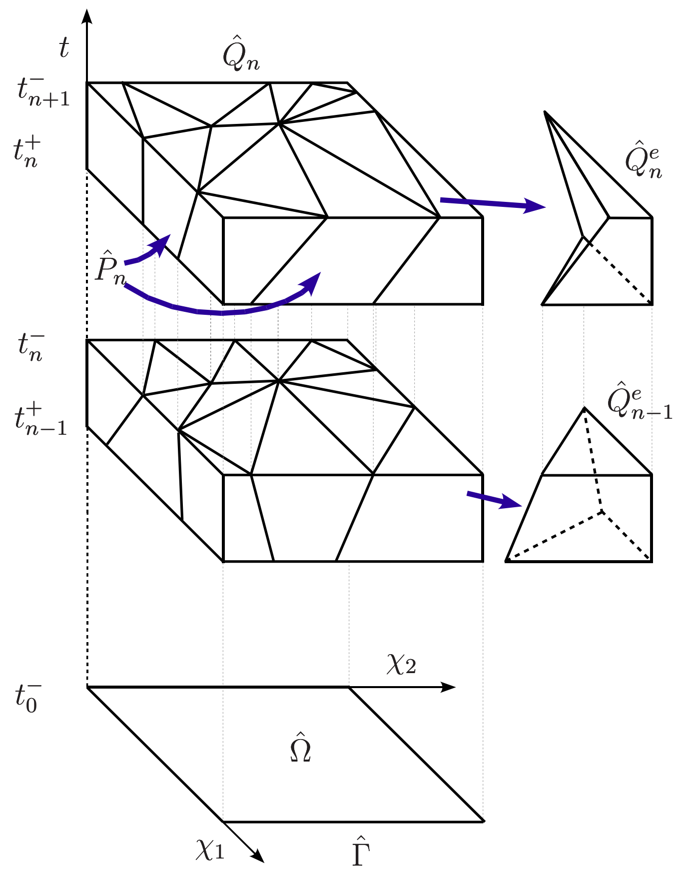

Figure 5.1. General finite element discretization in space-time.

with "lateral" boundary $\hat{P}_n = \hat{\Gamma} \times I_n$ ($\hat{\Gamma} = \partial\hat{\Omega}$ is the boundary of $\hat{\Omega}$). A sketch of the general discretization in space-time is presented in Figure 5.1. We will only make use of discretizations *prismatic* in time. The material domain $\hat{\Omega}$ is further divided into material-subdomains $\hat{\Omega}^e$ (elements in space, a partition of the initial configuration). Thus $\hat{\Omega} = \overline{\bigcup_{e=1}^{n_{el}} \hat{\Omega}^e}$, and, consequently, a typical space-time element is given by the prism (i.e., tensor product domain)

$$\hat{Q}_n^e = \hat{\Omega}^e \times I_n \tag{5.12}$$

It is also assumed that the space-time boundary is partitioned as $\hat{P}_n = \hat{P}_n^g \cup \hat{P}_n^h$, $\hat{P}_n^g \cap \hat{P}_n^h = \emptyset$ (i.e., $\hat{P}_n$ is divided into a Dirichlet boundary $\hat{P}_n^g$ and a Neumann boundary

$\hat{P}_n^h$). Let us define the test and trial function spaces as follows:

$$\hat{\mathcal{S}}^h = \Big\{ \hat{\boldsymbol{V}}^h : \hat{\boldsymbol{V}}^h \in (C^0(\hat{Q}))^m, \\ \hat{\boldsymbol{V}}^h\Big|_{\hat{Q}_n^e} \in (\mathcal{P}_1(\hat{\Omega}^e) \times \mathcal{P}_1(I_n))^m, \hat{\boldsymbol{V}}^h = \boldsymbol{G}_{bc}(t) \text{ on } \hat{P}_n^g \Big\} \tag{5.13}$$

$$\hat{\mathcal{V}}^h = \Big\{ \hat{\boldsymbol{W}}^h : \hat{\boldsymbol{W}}^h\Big|_{\hat{\Omega}} \in (C^0(\hat{\Omega}))^m, \\ \hat{\boldsymbol{W}}^h\Big|_{\hat{Q}_n^e} \in (\mathcal{P}_1(\hat{\Omega}^e) \times \mathcal{P}_0(I_n))^m, \hat{\boldsymbol{V}}^h = \boldsymbol{0} \text{ on } \hat{P}_n^g \Big\} \tag{5.14}$$

where $\boldsymbol{G}_{bc}(t)$ is the vector of Dirichlet boundary conditions, $\mathcal{P}_k$ is the set of polynomials up to degree k, and $m = n_d + 2$, $n_d \in \{1, 2, 3\}$. The trial function space $\hat{\mathcal{S}}^h$ is given by the piecewise-linear, continuous functions on $\hat{Q} = \hat{\Omega} \times]0, T[$, while the test function space $\hat{\mathcal{V}}^h$ is given by functions that are continuous piecewise-linear in space and discontinuous, piecewise-constant in time. The variational statement reads:

Find $\boldsymbol{Y}^h \in \hat{\mathcal{S}}^h$, such that $\forall \hat{\boldsymbol{W}}^h \in \hat{\mathcal{V}}^h$

$$\mathcal{B}(\hat{\boldsymbol{W}}, \boldsymbol{Y}^h) + \mathcal{SUPG}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h) + \mathcal{DC}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h) = \mathcal{F}(\hat{\boldsymbol{W}}) \tag{5.15}$$

with

$$\begin{aligned} \mathcal{B}(\hat{\boldsymbol{W}}, \boldsymbol{Y}^h) &= \int_{\hat{\Omega}} \hat{\boldsymbol{W}}^h(\boldsymbol{\chi}) \cdot \hat{\boldsymbol{U}}(\boldsymbol{Y}^h(\boldsymbol{\chi}, t_{n+1})) - \hat{\boldsymbol{W}}^h(\boldsymbol{\chi}) \cdot \hat{\boldsymbol{U}}(\boldsymbol{Y}^h(\boldsymbol{\chi}, t_n)) \; d\hat{\Omega} \\ &+ \int_{\hat{Q}_n} \Big(-\hat{\boldsymbol{W}}^h_{,i} \cdot \hat{\boldsymbol{G}}_i(\boldsymbol{Y}^h) + \hat{\boldsymbol{W}}^h \cdot \hat{\boldsymbol{Z}}(\boldsymbol{Y}^h) \Big) \; d\hat{Q} \\ &+ \int_{\hat{P}_n^g} \hat{\boldsymbol{W}}^h \cdot \hat{\boldsymbol{G}}_i(\boldsymbol{Y}^h) \hat{n}_i d\hat{P} \end{aligned} \tag{5.16}$$

$$\mathcal{F}(\hat{\boldsymbol{W}}) = -\int_{\hat{P}_n^h} \hat{\boldsymbol{W}}^h \cdot \hat{\boldsymbol{H}}_i \hat{n}_i \; d\hat{P} \tag{5.17}$$

$\hat{\boldsymbol{W}}$ is the vector-valued test function, $\hat{n}_i$ is the i-th component of the normal to the space-time boundary, and $\hat{\boldsymbol{H}}_i$ is the Neumann flux across the boundary in the i-th direction. The SUPG operator $\mathcal{SUPG}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h)$ will be defined subsequently. The discontinuity capturing operator $\mathcal{DC}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h)$, will be omitted in the following discussion, which applies to regions of smooth flows, away from discontinuities.

Remark 6 *The proposed formulation is second order in time and, following derivations analogous to [14], it can be easily proven to embed global conservation of mass, momentum and total energy.*

5.5 SUPG Stabilization

The SUPG stabilization operator can be abstractly defined as

$$\mathcal{SUPG}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h) \;=\; -\sum_{e=1}^{(n_{el})_n} \int_{\hat{Q}_n^e} (\hat{\mathcal{L}}^*_{SH}\hat{\boldsymbol{W}}_h) \cdot \hat{\boldsymbol{\tau}}\; \hat{\boldsymbol{Res}}(\boldsymbol{Y}^h) d\hat{Q} \tag{5.18}$$

where

$$\hat{\boldsymbol{Res}} \;=\; \hat{\mathcal{L}} = \hat{\boldsymbol{A}}_0\, \partial_t|_{\boldsymbol{\chi}} + \hat{\boldsymbol{A}}_i \partial_{\chi_i} + \hat{\boldsymbol{C}} \tag{5.19}$$

$$\hat{\mathcal{L}}_{SH} \;=\; \hat{\boldsymbol{A}}_0\, \partial_t|_{\boldsymbol{\chi}} + \hat{\boldsymbol{A}}_i \partial_{\chi_i} \tag{5.20}$$

$$\hat{\mathcal{L}}^*_{SH} \;=\; -\hat{\boldsymbol{A}}_0^T\, \partial_t|_{\boldsymbol{\chi}} - \hat{\boldsymbol{A}}_i^T \partial_{\chi_i} \tag{5.21}$$

$$\hat{\boldsymbol{\tau}} \;=\; \hat{\boldsymbol{\tau}}(\Delta t, h_e, \hat{\boldsymbol{A}}_0, \hat{\boldsymbol{A}}_i, \hat{\boldsymbol{C}}, \dots) \tag{5.22}$$

Δt is the time increment, and h_e is the e-th element mesh scale. A precise definition of $\hat{\boldsymbol{\tau}}$ is not needed for the purpose of the following discussion, and its functional dependence on the parameters and various terms in the formulation is sufficient to fully understand the issues under investigation.

Remark 7 *The rest of the discussion will be focused on assessing whether or not the perturbation to the Bubnov-Galerkin test function, $-(\hat{\mathcal{L}}^*_{SH}\hat{\boldsymbol{W}}_h) \cdot \hat{\boldsymbol{\tau}}$, is Galilean invariant.*

5.5.1 A multiscale view on Galilean invariance

Ideally, $\boldsymbol{Y}'$ is also an invariant of the Galilean transformation, since it is defined as the difference of $\boldsymbol{Y}$ and $\boldsymbol{Y}^h$ (see [13] for the trivial proof). However, the SUPG stabilization is obtained from a linearized multiscale decomposition of the Galerkin discretization, according to the following equations:

$$\mathcal{B}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h) \;+\; \int_{\hat{Q}_n} \hat{\mathcal{L}}^*_{SH}\hat{\boldsymbol{W}}^h \cdot \boldsymbol{Y}'\, d\hat{Q} = 0 \tag{5.23}$$

$$\boldsymbol{Y}' = \hat{\mathcal{L}}^{-1}_{SH}(-\hat{\boldsymbol{Res}}(\boldsymbol{Y}^h)) \;=\; -\int_{\hat{Q}_n} \hat{\boldsymbol{G}}'_{SH}\; \hat{\boldsymbol{Res}}(\boldsymbol{Y}^h)\, d\hat{Q}, \quad \text{in } \mathcal{V}'(\hat{Q}_n) \tag{5.24}$$

Let us assume for the moment that $\hat{\mathcal{L}}^*_{SH}$ in (5.23) and $\hat{\mathcal{L}}_{SH}$ in (5.24) are obtained using the full Fréchet derivative of the Galerkin residual. Because of the linearization, it is clear that both (5.23)–(5.24) may yield second-order Galilean inconsistencies. Consequently, when the *exact* solution $\boldsymbol{Y}'$ to the linearized subgrid problem (5.24) is inserted into the mesh-scale equation (5.23), only second-order Galilean inconsistencies should be generated.

If, however, the approximation to the Green's function operator $\hat{\boldsymbol{G}}'_{SH}$ is too coarse, $\boldsymbol{Y}' = -\hat{\boldsymbol{\tau}}(\boldsymbol{\chi}; t)\ \hat{\boldsymbol{Res}}(\boldsymbol{Y}^h(\boldsymbol{\chi}; t))$ may not retain low-order invariance properties. Since $\boldsymbol{Y}'$ is tied to the expression of the test function perturbation through $\hat{\boldsymbol{\tau}}$, stability can be at risk, as documented in [13]. The proposed extension to ALE equations of the approach developed in [14, 13] is designed to remove Galilean inconsistencies from the SUPG operator, with specific emphasis on the construction of the Petrov-Galerkin test space.

Remark 8 *In addition, it should be understood that an invariant approximation of* $\boldsymbol{Y}'$ *may be advisable, at least from the theoretical point of view, as will be further discussed in later sections.*

Chapter 6

Galilean invariance and the role of the subgrid-scale solution

Before undertaking an exhaustive discussion on the construction of the SUPG operator, it is important to understand how the numerical Galerkin residuals transform. A key point is the following:

Not all numerical residuals transform correctly, only the ones in advective form.

Let us review how the Euler equations of gas dynamics transform. Namely,

$$0 = \left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \frac{\partial \hat{J}\rho w_j}{\partial \chi_j} \tag{6.1}$$

$$0 = \left.\frac{\partial(\hat{J}\rho v_i)}{\partial t}\right|_{\boldsymbol{\chi}} + \frac{\partial}{\partial \chi_j}(\hat{J}\rho v_i w_j - \hat{P}_{ij}) - \rho \hat{J} g_i \tag{6.2}$$

$$0 = \left.\frac{\partial(\hat{J}\rho E)}{\partial t}\right|_{\boldsymbol{\chi}} + \frac{\partial}{\partial \chi_j}(\hat{J}\rho E w_j - v_i\hat{P}_{ij}) - \rho \hat{J} v_i g_i - \rho \hat{J} s \tag{6.3}$$

or, more simply,

$$0 = \hat{Res}^{\rho}(\rho;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \tag{6.4}$$

$$\begin{aligned} 0 &= \hat{Res}_i^{\rho\boldsymbol{v}}(\rho,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \\ &= v_i\ \hat{Res}^{\rho}(\rho;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) + \hat{Res}_i^{\boldsymbol{v}}(\rho,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \end{aligned} \tag{6.5}$$

$$\begin{aligned} 0 &= \hat{Res}^{E}(\rho,e,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \\ &= \Big(e + \frac{v_k v_k}{2}\Big)\hat{Res}^{\rho}(\rho;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \\ &\quad + v_i\ \hat{Res}_i^{\boldsymbol{v}}(\rho,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) + \hat{Res}^{e}(\rho,e,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \end{aligned} \tag{6.6}$$

where the mass conservation residual $\hat{Res}^{\rho}$, the momentum equation advective residual $\hat{Res}^{\boldsymbol{v}}$, and the internal energy equation residual $\hat{Res}^{e}$ are defined as

$$\hat{Res}^{\rho}(\rho;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) = \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}w_j\frac{\partial\rho}{\partial\chi_j} + \rho\,\mathrm{cof}\hat{F}_{ij}\frac{\partial v_i}{\partial\chi_j} \tag{6.7}$$

$$\hat{Res}_i^{\boldsymbol{v}}(\rho,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) = \hat{J}\rho\left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho w_j\frac{\partial v_i}{\partial\chi_j} + \frac{\partial p}{\partial\chi_j}\,\mathrm{cof}\hat{F}_{ij} - \hat{J}\rho g_i \tag{6.8}$$

$$\hat{Res}^{e}(\rho,e,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) = \hat{J}\rho\left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho w_j\frac{\partial e}{\partial\chi_j} + \frac{\partial v_i}{\partial\chi_j}p\,\mathrm{cof}\hat{F}_{ij} - \hat{J}\rho s \tag{6.9}$$

Here, the identity

$$\dot{\hat{J}} = \frac{\partial v_i}{\partial\chi_j}\mathrm{cof}\hat{F}_{ij} - \hat{J}\,\frac{\partial w_j}{\partial\chi_j} \tag{6.10}$$

has been used to rearrange the mass conservation equation (6.1) as follows:

$$\begin{aligned} 0 &= \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}w_j\frac{\partial\rho}{\partial\chi_j} + \hat{J}\rho\frac{\partial w_j}{\partial\chi_j} + \rho\left(\left.\frac{\partial\hat{J}}{\partial t}\right|_{\boldsymbol{\chi}} + w_j\frac{\partial\hat{J}}{\partial\chi_j}\right) \\ &= \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}w_j\frac{\partial\rho}{\partial\chi_j} + \hat{J}\rho\frac{\partial w_j}{\partial\chi_j} + \rho\dot{\hat{J}} \qquad (6.11) \\ &= \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}w_j\frac{\partial\rho}{\partial\chi_j} + \rho\frac{\partial v_i}{\partial\chi_j}\mathrm{cof}\hat{F}_{ij} \qquad (6.12) \end{aligned}$$

Recalling that $\boldsymbol{x}(\boldsymbol{X},t=0) = \boldsymbol{\chi}(\boldsymbol{X},t=0) = \boldsymbol{X}$, it easy to verify, that a Galilean transformation applied to the referential coordinate system reads

$$\begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{\chi}} \\ \tilde{\boldsymbol{w}} \\ \tilde{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} t \\ \boldsymbol{\chi} \\ \boldsymbol{w} \\ \boldsymbol{v}-\boldsymbol{V}^G \end{bmatrix}, \quad \text{or,} \quad \begin{bmatrix} t \\ \boldsymbol{\chi} \\ \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{\chi}} \\ \tilde{\boldsymbol{w}} \\ \tilde{\boldsymbol{v}}+\boldsymbol{V}^G \end{bmatrix} \tag{6.13}$$

Hence,

$$\hat{Res}^{\rho}(\rho;\boldsymbol{\chi},\boldsymbol{v},\boldsymbol{w},t) \xrightarrow{G} \hat{Res}^{\rho}(\rho;\tilde{\boldsymbol{\chi}},\tilde{\boldsymbol{v}},\tilde{\boldsymbol{w}},\tilde{t}) \tag{6.14}$$

$$\hat{Res}_i^{\boldsymbol{v}}(\rho,p;\boldsymbol{\chi},\boldsymbol{v},\boldsymbol{w},t) \xrightarrow{G} \hat{Res}_i^{\tilde{\boldsymbol{v}}}(\rho,p;\tilde{\boldsymbol{\chi}},\tilde{\boldsymbol{v}},\tilde{\boldsymbol{w}},\tilde{t}) \tag{6.15}$$

$$\hat{Res}^{e}(\rho,e,p;\boldsymbol{\chi},\boldsymbol{v},\boldsymbol{w},t) \xrightarrow{G} \hat{Res}^{e}(\rho,e,p;\tilde{\boldsymbol{\chi}},\tilde{\boldsymbol{v}},\tilde{\boldsymbol{w}},\tilde{t}) \tag{6.16}$$

$$\begin{aligned}
\hat{Res}_i^{\rho \boldsymbol{v}}(\rho, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) \;\xrightarrow{G}\; & \tilde{v}_i \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + \hat{Res}_i^{\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + V_i^G \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t})
\end{aligned} \tag{6.17}$$

$$\begin{aligned}
\hat{Res}^{E}(\rho, e, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) \;\xrightarrow{G}\; & \left(e + \frac{\tilde{v}_k \tilde{v}_k}{2}\right) \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + \tilde{v}_i \, \hat{Res}_i^{\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + \hat{Res}^{e}(\rho, e, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + V_i^G \Big(\tilde{v}_i \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& \qquad + \hat{Res}_i^{\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \Big) \\
& + \frac{V_k^G V_k^G}{2} \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t})
\end{aligned} \tag{6.18}$$

The last two equations can be written more compactly as:

$$\begin{aligned}
\hat{Res}_i^{\rho \boldsymbol{v}}(\rho, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) \;\xrightarrow{G}\; & \hat{Res}_i^{\rho \tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + V_i^G \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t})
\end{aligned} \tag{6.19}$$

$$\begin{aligned}
\hat{Res}^{E}(\rho, e, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) \;\xrightarrow{G}\; & \hat{Res}^{E}(\rho, e, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + V_i^G \hat{Res}_i^{\rho \tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + \frac{V_k^G V_k^G}{2} \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t})
\end{aligned} \tag{6.20}$$

Therefore, as expected, the equations transform appropriately, because the terms multiplied by the transformation velocity $\boldsymbol{V}^G$ annihilate exactly. In other words, if an exact multiscale decomposition of the solution is applied, the resulting equations would satisfy the invariance principle.

If, however, as already mentioned in [13], the subgrid-scale problem is solved only approximately, the situation is different and the numerical residuals are not necessarily invariant. On the one hand, no matter the numerical quadrature used, the numerical approximations to the "advective" residuals $\hat{Res}^{h;\rho}$, $\hat{\boldsymbol{Res}}^{h;\boldsymbol{v}}$, and $\hat{Res}^{h;e}$ would transform correctly, because the velocity $\boldsymbol{v}$ appears only in differentiated form inside their expressions. On the other hand, $\hat{Res}^{h;\rho\boldsymbol{v}}$ and $\hat{Res}^{h;E}$ would not transform correctly, because $\boldsymbol{V}^G$ multiplies some of the non-vanishing residual terms. Hence, the approximation to the subgrid-scale solution $\boldsymbol{Y}' \approx -\hat{\boldsymbol{\tau}} \; \hat{\boldsymbol{Res}}(\boldsymbol{Y}^h)$ would not be invariant if these residuals were used in its construction.

It is also important to realize, however, that residuals are usually higher-order corrections: In the computations performed in [14, 13], virtually no difference in the results was observed between SUPG operators with and without invariant residuals in the approximation to $\boldsymbol{Y}'$. The fact that instabilities were experienced only for a non-invariant SUPG test function perturbation indicates that the latter is far more stringent then Galilean consistency of the residual terms. Nonetheless, it should be advisable to preserve invariance also for the approximations to $\boldsymbol{Y}'$.

Chapter 7

Quasi-linear form of the ALE equations and invariance

Quasi-linear differential forms of the ALE equations have a central role in the design of SUPG stabilization operators. A quasi-linear form of the Euler equations using pressure variables will be derived using the traditional Fréchet differentiation approach and the new minimal approach of [13]. The structure and invariance properties of the resulting SUPG perturbations of the Galerkin test function will be analyzed.

For simplicity, heat fluxes are assumed absent. Applying the Piola identity

$$\frac{\partial(\mathrm{cof}\hat{F}_{ij})}{\partial\chi_j} \equiv \frac{\partial}{\partial\chi_j}\left(\hat{J}\hat{F}_{ji}^{-1}\right) = 0 \tag{7.1}$$

to the stress terms in (6.2)–(6.3) yields

$$\begin{aligned} \frac{\partial\hat{P}_{ij}}{\partial\chi_j} &= \frac{\partial(\hat{J}\sigma_{ik}\hat{F}_{jk}^{-1})}{\partial\chi_j} = \frac{\partial\sigma_{ik}}{\partial\chi_j}\hat{J}\hat{F}_{jk}^{-1} = -\frac{\partial p}{\partial\chi_j}\hat{J}\hat{F}_{ji}^{-1} = -\frac{\partial p}{\partial\chi_j}\mathrm{cof}\hat{F}_{ij} && (7.2)\\ \frac{\partial v_i\hat{P}_{ij}}{\partial\chi_j} &= \frac{\partial(v_i\hat{J}\sigma_{ik}\hat{F}_{jk}^{-1})}{\partial\chi_j} = \frac{\partial(v_i\sigma_{ik})}{\partial\chi_j}\hat{J}\hat{F}_{jk}^{-1} = -\frac{\partial(v_i p)}{\partial\chi_j}\mathrm{cof}\hat{F}_{ij} && (7.3) \end{aligned}$$

Using (6.10),

$$0 = \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}w_j\frac{\partial\rho}{\partial\chi_j} + \rho\,\mathrm{cof}\hat{F}_{ij}\frac{\partial v_i}{\partial\chi_j} \tag{7.4}$$

$$
\begin{aligned}
0 &= v_i \left(\hat{J}\, \frac{\partial \rho}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial \rho}{\partial \chi_j} + \rho \operatorname{cof}\hat{F}_{ij} \frac{\partial v_i}{\partial \chi_j} \right) \\
&\quad + \hat{J}\rho\, \frac{\partial v_i}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J}\rho w_j \frac{\partial v_i}{\partial \chi_j} + \frac{\partial p}{\partial \chi_j} \operatorname{cof}\hat{F}_{ij} - \hat{J}\rho g_i
\end{aligned}
\tag{7.5}
$$

$$
\begin{aligned}
0 &= \left(e + \frac{v_k v_k}{2} \right) \left(\hat{J} \frac{\partial \rho}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial \rho}{\partial \chi_j} + \rho \operatorname{cof}\hat{F}_{ij} \frac{\partial v_i}{\partial \chi_j} \right) \\
&\quad + \left(\hat{J}\, \frac{\partial \rho}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial \rho}{\partial \chi_j} + \rho \operatorname{cof}\hat{F}_{ij} \frac{\partial v_i}{\partial \chi_j} \right) \\
&\quad + v_i \left(\hat{J}\rho\, \frac{\partial v_i}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J}\rho w_j \frac{\partial v_i}{\partial \chi_j} + \frac{\partial p}{\partial \chi_j} \operatorname{cof}\hat{F}_{ij} - \hat{J}\rho g_i \right) \\
&\quad + \hat{J}\rho\, \frac{\partial e}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J}\rho w_j \frac{\partial e}{\partial \chi_j} + \frac{\partial v_i}{\partial \chi_j} p \operatorname{cof}\hat{F}_{ij} - \hat{J}\rho s
\end{aligned}
\tag{7.6}
$$

Remark 9 *The momentum and energy equations contain the mass conservation equation multiplied by the velocity component v_i, and internal energy e, respectively. In addition, the energy equation contains the kinetic-energy equation, the scalar product of the velocity times the momentum equation.*

In the present case, the following identity will become very useful:

$$
\begin{aligned}
\hat{J}\, \frac{\partial e}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial e}{\partial \chi_j} &= \hat{J}\, \frac{\partial e}{\partial p}\bigg|_{\rho} \left(\frac{\partial p}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial p}{\partial \chi_j} \right) + \hat{J}\, \frac{\partial e}{\partial \rho}\bigg|_{p} \left(\frac{\partial \rho}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial \rho}{\partial \chi_j} \right) \\
&= \hat{J} e_{,p} \left(\frac{\partial p}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial p}{\partial \chi_j} \right) + \hat{J} e_{,\rho} \left(\frac{\partial \rho}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial \rho}{\partial \chi_j} \right) \\
&= \hat{J} e_{,p} \left(\frac{\partial p}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial p}{\partial \chi_j} \right) - e_{,\rho} \rho \operatorname{cof}\hat{F}_{ij} \frac{\partial v_i}{\partial \chi_j}
\end{aligned}
\tag{7.7}
$$

where (6.1) has been used in the last step. Here, $e_{,p} = \frac{\partial e}{\partial p}\Big|_{\rho}$, and $e_{,\rho} = \frac{\partial e}{\partial \rho}\Big|_{p}$.

7.1 The "standard", non-invariant approach

The quasilinear vector form reads

$$
\hat{\boldsymbol{A}}_0\, \partial_t|_{\boldsymbol{\chi}} \boldsymbol{Y} + \hat{\boldsymbol{A}}_i(\boldsymbol{Y})\, \partial_{\chi_i} \boldsymbol{Y} + \hat{\boldsymbol{C}}(\boldsymbol{Y})\, \boldsymbol{Y} = \boldsymbol{0}
\tag{7.8}
$$

with

$$
\hat{\boldsymbol{A}}_0^{(NG)} = \begin{bmatrix} \hat{J} & 0 & 0 & 0 & 0 \\ \hat{J}v_1 & \hat{J}\rho & 0 & 0 & 0 \\ \hat{J}v_2 & 0 & \hat{J}\rho & 0 & 0 \\ \hat{J}v_3 & 0 & 0 & \hat{J}\rho & 0 \\ \hat{J}(E+\rho e_{,\rho}) & \hat{J}\rho v_1 & \hat{J}\rho v_2 & \hat{J}\rho v_3 & \hat{J}\rho e_{,p} \end{bmatrix}, \tag{7.9}
$$

$$
\hat{\boldsymbol{C}}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\hat{J}g_1 & 0 & 0 & 0 & 0 \\ -\hat{J}g_2 & 0 & 0 & 0 & 0 \\ -\hat{J}g_3 & 0 & 0 & 0 & 0 \\ -\hat{J}s & -\hat{J}\rho g_1 & -\hat{J}\rho g_2 & -\hat{J}\rho g_3 & 0 \end{bmatrix}, \tag{7.10}
$$

and, for $i = 1, 2, 3$,

$$
\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{l|l|l|l|l} \hat{J}w_i & \rho \operatorname{cof}\hat{F}_{1i} & \rho \operatorname{cof}\hat{F}_{2i} & \rho \operatorname{cof}\hat{F}_{3i} & 0 \\ \hline \hat{J}w_i v_1 & \hat{J}\rho w_i + \rho \operatorname{cof}\hat{F}_{1i} v_1 & \rho \operatorname{cof}\hat{F}_{2i} v_1 & \rho \operatorname{cof}\hat{F}_{3i} v_1 & \operatorname{cof}\hat{F}_{1i} \\ \hline \hat{J}w_i v_2 & \rho \operatorname{cof}\hat{F}_{1i} v_2 & \hat{J}\rho w_i + \rho \operatorname{cof}\hat{F}_{2i} v_2 & \rho \operatorname{cof}\hat{F}_{3i} v_2 & \operatorname{cof}\hat{F}_{2i} \\ \hline \hat{J}w_i v_3 & \rho \operatorname{cof}\hat{F}_{1i} v_3 & \rho \operatorname{cof}\hat{F}_{2i} v_3 & \hat{J}\rho w_i + \rho \operatorname{cof}\hat{F}_{3i} v_3 & \operatorname{cof}\hat{F}_{3i} \\ \hline \hat{J}w_i E & \hat{J}\rho w_i v_1 + (\rho E + p - \rho^2 e_{,\rho}) \operatorname{cof}\hat{F}_{1i} & \hat{J}\rho w_i v_2 + (\rho E + p - \rho^2 e_{,\rho}) \operatorname{cof}\hat{F}_{2i} & \hat{J}\rho w_i v_3 + (\rho E + p - \rho^2 e_{,\rho}) \operatorname{cof}\hat{F}_{3i} & \hat{J}\rho w_i e_{,p} + \operatorname{cof}\hat{F}_{ki} v_k \end{array} \right] \tag{7.11}
$$

Remark 10 *This choice leads to Jacobians of the Euler fluxes which are not invariant if considered separately. By inspection, it is easy to realize that there is a large number of terms which contain components of the velocity vector* $\boldsymbol{v}$*. Therefore, a single Euler flux Jacobian or an arbitrary combination of Euler flux Jacobians are not necessarily invariant. This is precisely the situation for the perturbation to the test function* $-(\hat{\boldsymbol{\mathcal{L}}}^*_{SH}\hat{\boldsymbol{W}}^h) \cdot \hat{\boldsymbol{\tau}} = (\hat{\boldsymbol{A}}_0^T\, \partial_t|_{\boldsymbol{\chi}} \hat{\boldsymbol{W}}^h + \hat{\boldsymbol{A}}_i^T \partial_{\chi_i} \hat{\boldsymbol{W}}^h)\hat{\boldsymbol{\tau}}$*, which lacks invariance properties, with potentially very negative consequences on the overall stability of the formulation.*

Remark 11 *Although in principle it is possible to develop a tensor* $\hat{\boldsymbol{\tau}}$ *which would make the perturbation to the test function Galilean invariant, it should be evident to the reader that, in practice, the current structure of the Jacobians makes this task extremely difficult.*

Remark 12 *Obviously, also the approximation to* $\boldsymbol{Y}'$ *is not invariant.*

7.2 A new Galilean invariant approach

The previous approach is not the only way to derive a quasi-linear form of the Euler equations. Starting from (7.4)–(7.6), additional algebraic manipulations can be performed. Let us therefore remove the mass conservation equation terms from the momentum and total energy equations, and the kinetic energy equation from the total energy equation. Hence,

$$0 = \hat{J}\left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial \rho}{\partial \chi_j} + \rho \operatorname{cof} \hat{F}_{ij} \frac{\partial v_i}{\partial \chi_j} \tag{7.12}$$

$$0 = \hat{J}\rho\left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho w_j \frac{\partial v_i}{\partial \chi_j} + \frac{\partial p}{\partial \chi_j} \operatorname{cof} \hat{F}_{ij} - \hat{J}\rho g_i \tag{7.13}$$

$$0 = \hat{J}\rho\left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho w_j \frac{\partial e}{\partial \chi_j} + \frac{\partial v_i}{\partial \chi_j} p \operatorname{cof} \hat{F}_{ij} - \hat{J}\rho s \tag{7.14}$$

Thus,

$$\hat{\boldsymbol{A}}_0^{(Gal)} = \begin{bmatrix} \hat{J} & 0 & 0 & 0 & 0 \\ 0 & \hat{J}\rho & 0 & 0 & 0 \\ 0 & 0 & \hat{J}\rho & 0 & 0 \\ 0 & 0 & 0 & \hat{J}\rho & 0 \\ 0 & 0 & 0 & 0 & \hat{J}\rho e_{,p} \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\hat{J}g_1 & 0 & 0 & 0 & 0 \\ -\hat{J}g_2 & 0 & 0 & 0 & 0 \\ -\hat{J}g_3 & 0 & 0 & 0 & 0 \\ -\hat{J}s & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{7.15}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(Gal)} = \left[\begin{array}{c|c|c|c|c} \hat{J}w_i & \rho \operatorname{cof}\hat{F}_{1i} & \rho \operatorname{cof}\hat{F}_{2i} & \rho \operatorname{cof}\hat{F}_{3i} & 0 \\ \hline 0 & \hat{J}\rho w_i & 0 & 0 & \operatorname{cof}\hat{F}_{1i} \\ \hline 0 & 0 & \hat{J}\rho w_i & 0 & \operatorname{cof}\hat{F}_{2i} \\ \hline 0 & 0 & 0 & \hat{J}\rho w_i & \operatorname{cof}\hat{F}_{3i} \\ \hline 0 & (p-\rho^2 e_{,\rho})\operatorname{cof}\hat{F}_{1i} & (p-\rho^2 e_{,\rho})\operatorname{cof}\hat{F}_{2i} & (p-\rho^2 e_{,\rho})\operatorname{cof}\hat{F}_{3i} & \hat{J}\rho w_i e_{,p} \end{array}\right] \tag{7.16}$$

Remark 13 *Each of the generalized advective matrices developed respects the principle of Galilean invariance, since they are function of $\hat{\boldsymbol{F}}$, $\hat{J}$, $\boldsymbol{w}$, p, ρ, $e_{,p}$, and $e_{,\rho}$, all invariant quantities.*

Remark 14 *One can think about the proposed approach as being "minimalist". In fact it produces the minimal number of entries in the Jacobians while still retaining the generalized advective structure of the quasi-linear form, now reduced to the mass conservation equation, the advective form of the momentum equation, and the advective form of the internal energy equation.*

Remark 15 *With respect to the standard Jacobians, the Galilean invariant Jacobians require about 77 fewer terms to be computed. The compact sparsity pattern is clearly noticeable in (7.15)–(7.16). It is unusual and quite remarkable that a consistent approach leads to a significant reduction in the computational cost.*

7.3 The Lagrangian limit

The Lagrangian limit is very instructive in understanding the issues related to lack of Galilean invariance.

7.3.1 Standard, non-invariant approach

$$\hat{\boldsymbol{A}}_0^{(NG)} = \begin{bmatrix} J & 0 & 0 & 0 & 0 \\ Jv_1 & \rho_0 & 0 & 0 & 0 \\ Jv_2 & 0 & \rho_0 & 0 & 0 \\ Jv_3 & 0 & 0 & \rho_0 & 0 \\ J(E+\rho e_{,\rho}) & \rho_0 v_1 & \rho_0 v_2 & \rho_0 v_3 & \rho_0 e_{,p} \end{bmatrix}, \tag{7.17}$$

$$\hat{\boldsymbol{C}}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -Jg_1 & 0 & 0 & 0 & 0 \\ -Jg_2 & 0 & 0 & 0 & 0 \\ -Jg_3 & 0 & 0 & 0 & 0 \\ -Js & -\rho_0 g_1 & -\rho_0 g_2 & -\rho_0 g_3 & 0 \end{bmatrix}, \tag{7.18}$$

where $\rho_0 = \rho\ J$ is the initial density distribution, and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{c|l|l|l|l} 0 & \rho\ \mathrm{cof}F_{1i} & \rho\ \mathrm{cof}F_{2i} & \rho\ \mathrm{cof}F_{3i} & 0 \\ \hline 0 & \rho\ \mathrm{cof}F_{1i}v_1 & \rho\ \mathrm{cof}F_{2i}v_1 & \rho\ \mathrm{cof}F_{3i}v_1 & \mathrm{cof}F_{1i} \\ \hline 0 & \rho\ \mathrm{cof}F_{1i}v_2 & \rho\ \mathrm{cof}F_{2i}v_2 & \rho\ \mathrm{cof}F_{3i}v_2 & \mathrm{cof}F_{2i} \\ \hline 0 & \rho\ \mathrm{cof}F_{1i}v_3 & \rho\ \mathrm{cof}F_{2i}v_3 & \rho\ \mathrm{cof}F_{3i}v_3 & \mathrm{cof}F_{3i} \\ \hline 0 & (\rho E+p & (\rho E+p & (\rho E+p & \mathrm{cof}\hat{F}_{ki}v_k \\ & \ -\rho^2 e_{,\rho})\mathrm{cof}\hat{F}_{1i} & \ -\rho^2 e_{,\rho})\mathrm{cof}\hat{F}_{2i} & \ -\rho^2 e_{,\rho})\mathrm{cof}\hat{F}_{3i} & \end{array}\right] \tag{7.19}$$

Remark 16 *A large number of terms are multiplied by the velocity components, with the potential for very dangerous consequences, since now a standard implementation of the SUPG operator would generate nodal forces depending on the observer. As documented in [13] and Figure 1.1, simulations of even mild shocks could not be successfully completed with this approach.*

7.3.2 Galilean invariant approach

$$\hat{\boldsymbol{A}}_0^{(Gal)} = \begin{bmatrix} J & 0 & 0 & 0 & 0 \\ 0 & \rho_0 & 0 & 0 & 0 \\ 0 & 0 & \rho_0 & 0 & 0 \\ 0 & 0 & 0 & \rho_0 & 0 \\ 0 & 0 & 0 & 0 & \rho_0 e_{,p} \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -Jg_1 & 0 & 0 & 0 & 0 \\ -Jg_2 & 0 & 0 & 0 & 0 \\ -Jg_3 & 0 & 0 & 0 & 0 \\ -Js & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.20}$$

where $\rho_0 = \rho\, J$, and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(Gal)} = \begin{bmatrix} 0 & \rho\,\mathrm{cof}F_{1i} & \rho\,\mathrm{cof}F_{2i} & \rho\,\mathrm{cof}F_{3i} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{cof}F_{1i} \\ 0 & 0 & 0 & 0 & \mathrm{cof}F_{2i} \\ 0 & 0 & 0 & 0 & \mathrm{cof}F_{3i} \\ 0 & (p-\rho^2 e_{,\rho})\mathrm{cof}F_{1i} & (p-\rho^2 e_{,\rho})\mathrm{cof}F_{2i} & (p-\rho^2 e_{,\rho})\mathrm{cof}F_{3i} & 0 \end{bmatrix} \tag{7.21}$$

Remark 17 *None of the entries in (7.20)–(7.21) depends on $\boldsymbol{v}$. The Lagrangian formulation presented here is slightly different from the one in [14], in which the algebraic constraint $\rho_0 = \rho\, J$ is enforced strongly in the equations. This amounts to removing the first row and column from (7.20)–(7.21). Using a diagonal $\hat{\boldsymbol{\tau}}$ tensor, the simulations in [14] never suffered from instabilities, even in the most demanding implosion computations, with shock strengths exceeding Mach 10,000,000.*

7.3.3 A simple example: one-dimensional gas dynamics

With a simple example we will show the effect of lack of Galilean invariance on the perturbation to the test function. Let us consider the general definition of the $\hat{\boldsymbol{\tau}}$ tensor given in [15]:

$$\hat{\boldsymbol{\tau}} = \boldsymbol{A}_0^{-1} \left(\boldsymbol{C}^2 + \left(\frac{\partial \xi_0}{\partial t} \right)^2 \boldsymbol{I} + \frac{\partial \xi_i}{\partial X_j} \frac{\partial \xi_i}{\partial X_k} \boldsymbol{A}_j \boldsymbol{A}_k \right)^{-1/2} \tag{7.22}$$

where $\boldsymbol{A}_1 = \hat{\boldsymbol{A}}_1 \boldsymbol{A}_0^{-1}$ and ξ_i are the coordinates in the parent domain of each element, and ξ_0 refers to the time axis.

Remark 18 *It is not the intention of the author to critisize a particular reference in the available litarature. The large majority of stabilization operators adopts similar expressions, so it is fair to say that the following example is prototypical of current stabilization approaches to compressible flows.*

For an ideal gas in one dimension,

$$\frac{\partial \xi_0}{\partial t} = \frac{2}{\Delta t} \tag{7.23}$$

$$\frac{\partial \xi_i}{\partial X_j}\frac{\partial \xi_i}{\partial X_k}\boldsymbol{A}_j\boldsymbol{A}_k = \left(\frac{2}{\Delta X}\right)^2 \boldsymbol{A}_1^2 \tag{7.24}$$

Substituting (7.17)–(7.19) into (7.22), and recalling that in the current space-time formulation $\partial_t|_{\boldsymbol{\chi}}\hat{\boldsymbol{W}} = \boldsymbol{0}$, it can be obtained:

$$\begin{aligned} -(\hat{\mathcal{L}}^*_{SH}\hat{\boldsymbol{W}}^h)\cdot\hat{\boldsymbol{\tau}} &= (\hat{\boldsymbol{A}}_0^T\,\partial_t|_{\boldsymbol{\chi}}\hat{\boldsymbol{W}}^h + \hat{\boldsymbol{A}}_i^T\partial_{\chi_i}\hat{\boldsymbol{W}}^h)\hat{\boldsymbol{\tau}} \\ &= \partial_X\hat{\boldsymbol{W}}^h\hat{\boldsymbol{A}}_1\hat{\boldsymbol{\tau}} \end{aligned} \tag{7.25}$$

with

$$\hat{\boldsymbol{A}}_1\hat{\boldsymbol{\tau}} = \begin{pmatrix} -\frac{v}{J}\beta & \frac{\beta}{J} & 0 \\ \frac{v^2(-3+\gamma)\beta}{2J} & \frac{-v(-2+\gamma)\beta}{J} & \frac{(\gamma-1)\beta}{J} \\ \frac{v\left(-2p\gamma+v^2\left(2-3\gamma+\gamma^2\right)\rho\right)\beta}{2(\gamma-1)\rho J} & \frac{\left(2p\gamma+v^2\left(-3+5\gamma-2\gamma^2\right)\rho\right)\beta}{2(\gamma-1)\rho J} & \frac{v(\gamma-1)\beta}{J} \end{pmatrix} \tag{7.26}$$

$$\beta = \frac{\Delta t}{2\sqrt{1+\alpha^2}}, \quad \alpha = \frac{c_s\Delta t}{\Delta x}, \quad \Delta x = J\Delta X, \quad c_s = \sqrt{\gamma\frac{p}{\rho}} \tag{7.27}$$

Depending on the value of v, the perturbation to the test function can assume a wide range of values (v can also be negative, so that sign inversions can occur, particularly problematic to stability). It is clear that this approach leads to observer-dependent stabilization operators. Instead, in the case of the Galiean invariant approach,

$$\hat{\boldsymbol{A}}_1\hat{\boldsymbol{\tau}} = \begin{pmatrix} 0 & \frac{\beta}{J} & 0 \\ 0 & 0 & \frac{\gamma-1}{J}\beta \\ 0 & \frac{\beta c_s^2}{(\gamma-1)J} & 0 \end{pmatrix} \tag{7.28}$$

independent of the velocity v.

7.4 The Eulerian limit

The standard approach is widely documented in the literature for Eulerian meshes, and will be shown to be inconsistent with the Galilean principle.

7.4.1 Standard, non-invariant approach

$$\hat{\boldsymbol{A}}_0^{(NG)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & \rho & 0 & 0 & 0 \\ v_2 & 0 & \rho & 0 & 0 \\ v_3 & 0 & 0 & \rho & 0 \\ (E+\rho e_{,\rho}) & \rho v_1 & \rho v_2 & \rho v_3 & \rho e_{,p} \end{bmatrix}, \tag{7.29}$$

$$\hat{\boldsymbol{C}}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 & 0 & 0 & 0 & 0 \\ -g_2 & 0 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & 0 & 0 \\ -s & -\rho g_1 & -\rho g_2 & -\rho g_3 & 0 \end{bmatrix}, \tag{7.30}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{l|l|l|l|l} v_i & \rho\delta_{1i} & \rho\delta_{2i} & \rho\delta_{3i} & 0 \\ \hline v_i v_1 & \rho(v_i+\delta_{1i}v_1) & \rho\delta_{2i}v_1 & \rho\delta_{3i}v_1 & \delta_{1i} \\ \hline v_i v_2 & \rho\delta_{1i}v_2 & \rho(v_i+\delta_{2i}v_2) & \rho\delta_{3i}v_2 & \delta_{2i} \\ \hline v_i v_3 & \rho\delta_{1i}v_3 & \rho\delta_{2i}v_3 & \rho(v_i+\delta_{3i}v_3) & \delta_{3i} \\ \hline v_i(E+\rho e_{,\rho}) & \rho v_i v_1 + & \rho v_i v_2 + & \rho v_i v_3 + & \rho v_i e_{,p} + v_i \\ & (\rho E+p)\delta_{1i} & (\rho E+p)\delta_{2i} & (\rho E+p)\delta_{3i} & \end{array}\right] \tag{7.31}$$

Remark 19 *Although non-invariant, the previous Jacobians and their variations with different sets of variables are currently used in the large majority of SUPG-stabilized finite element methods for compressible flow applications, with potentially very dangerous consequences on the reliability of the results.*

Remark 20 *As a justification for the inconsistencies found in the literature to date, it is virtually impossible to discern whether a velocity term transforms correctly, if only the Eulerian form of the equations is available, since* $\boldsymbol{w} = \hat{\boldsymbol{F}}^{-1}(\boldsymbol{v} - \hat{\boldsymbol{v}}) = \boldsymbol{I}(\boldsymbol{v} - \boldsymbol{0}) = \boldsymbol{v}$. *The reverse approach is needed, in which first a consistent ALE formulation is developed and then the Eulerian equations are derived as a limit.*

Remark 21 *It is interesting to observe that there is at least one formulation in the literature of SUPG methods for compressible flows which respects the Galilean invariance principle. In fact, the first example of application of SUPG methods to the compressible Euler equations on fixed meshes, as presented by Hughes and Tezduyar [12], does possess invariant properties with respect to Galilean shifts.*

7.4.2 Galilean invariant approach

$$\hat{\boldsymbol{A}}_0^{(Gal)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & \rho e_{,p} \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 & 0 & 0 & 0 & 0 \\ -g_2 & 0 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & 0 & 0 \\ -s & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.32}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(Gal)} = \begin{bmatrix} v_i & \rho\delta_{1i} & \rho\delta_{2i} & \rho\delta_{3i} & 0 \\ 0 & \rho v_i & 0 & 0 & \delta_{1i} \\ 0 & 0 & \rho v_i & 0 & \delta_{2i} \\ 0 & 0 & 0 & \rho v_i & \delta_{3i} \\ 0 & (p-\rho^2 e_{,\rho})\delta_{1i} & (p-\rho^2 e_{,\rho})\delta_{2i} & (p-\rho^2 e_{,\rho})\delta_{3i} & \rho v_i e_{,p} \end{bmatrix} \tag{7.33}$$

Remark 22 *The discussion presented so far has been in the context of pressure variables, but very similar conclusions can be drawn for virtually any other set of variables. Therefore, the analysis developed has a very general applicability.*

Chapter 8

Summary

An extended invariance analysis of stabilized methods for compressible and incompressible flows has been presented in the general ALE context. It was shown that most of the stabilization operators designed to date for compressible flow applications on Eulerian meshes *do not* satisfy the principle of Galilean invariance, with the exception of the early approach of Hughes and Tezduyar [12]. It has been argued that this is both a physical and numerical flaw, since a non-invariant Petrov-Galerkin test space can have direct consequences on the stability properties of SUPG methods.

Given the disastrous results documented in [13] for the Lagrangian limit, it is the opinion of the author that *at least* a "reasonable doubt" on the correctness of the stabilization techniques lacking invariance has to be raised. It was shown that a simple manipulation of the quasi-linear form of the equations of motion leads to Galilean invariant formulations, which also have the advantage of a significant reduction in the computational cost of the stabilization. The reliability of the new approach under severe conditions was proven in [14], where diagonal $\boldsymbol{\tau}$ tensors were used to stabilize shocks of strength in excess of Mach 10,000,000.

More work is needed to find additional examples in which lack of Galilean invariance leads to catastrophic results, to broaden the discussion and further confirm the importance of the issue.

As a final comment, when considering complex compressible flow applications, conformity with physics principles in the design of stabilization and subgrid-scale operators appears to be one of the *not so many* guidelines available to the scientist. In this context, the price to be paid by neglecting the *Galilean sanity check* may be much greater than expected. The effects of invariance inconsistencies are usually difficult to isolate and track *a posteriori*, in large-scale industrial implementations.

References

[1] T. Belytschko, W. K. Liu, and B. Moran. *Nonlinear Finite Elements for Continua and Structures*. J. Wiley & Sons, New York, 2000.

[2] A. N. Brooks and T. J. R. Hughes. Streamline upwind / Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, **32**:199–259, 1982.

[3] J. Donea. Arbitrary lagrangian-eulerian finite element methods. In T. Belytschko and T. J. R. Hughes, editors, *Computational Methods in Transient Analysis*. Elsevier, 1983.

[4] J. Donea, A. Huerta, J. Ph. Ponthot, and A. Rodríguez-Ferranand. Arbitrary Lagrangian-Eulerian methods. In R. Stein E., de Borst and T.J.R. Hughes, editors, *Encyclopedia of Computational Mechanics*. John Wiley & Sons, 2004.

[5] T. J. R. Hughes, V. M. Calo, and G. Scovazzi. Variational and multiscale methods in turbulence. In *Proceedings of the XXI International Congress of Theoretical and Applied Mechanics*. IUTAM, Kluwer, 2004.

[6] T. J. R. Hughes, L. Mazzei, and K. E. Jansen. Large eddy simulation and the variational multiscale method. *Computing and Visualization in Science*, **3**(47):147–162, 2000.

[7] T. J. R. Hughes, L. Mazzei, A. A. Oberai, and A.A. Wray. The multiscale formulation of large eddy simulation: decay of homogenous isotropic turbulence. *Physics of Fluids*, **13**:505–512, 2001.

[8] T. J. R. Hughes, A. A. Oberai, and L. Mazzei. Large eddy simulation of turbulent channel flows by the variational multiscale method. *Physics of Fluids*, **13**(6):1784–1799, 2001.

[9] T.J.R. Hughes. Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid-scale models, bubbles and the origin of stabilized methods. *Computer Methods in Applied Mechanics and Engineering*, **127**:387–401, 1995.

[10] T.J.R. Hughes, G.R. Feijóo, L. Mazzei, and J.-B. Quincy. The Variational Multiscale Method — a paradigm for computational mechanics. *Computer Methods in Applied Mechanics and Engineering*, **166**:3–24, 1998.

[11] T.J.R. Hughes, G. Scovazzi, and Franca L.P. Multiscale and stabilized methods. In R. Stein E., de Borst and T.J.R. Hughes, editors, *Encyclopedia of Computational Mechanics*. John Wiley & Sons, 2004.

[12] T.J.R. Hughes and T. E. Tezduyar. Finite element methods for first-order hyperbolic system with particular emphasis on the compressible Euler equations. *Computer Methods in Applied Mechanics and Engineering*, **45**:217–284, 1984.

[13] G. Scovazzi. Stabilized shock hydrodynamics: II. Design and physical interpretation of the SUPG operator for Lagrangian computations. *Computer Methods in Applied Mechanics and Engineering*, 2005. Submitted.

[14] G. Scovazzi, M. A. Christon, T. J. R. Hughes, and J. N. Shadid. Stabilized shock hydrodynamics: I. A Lagrangian method. *Computer Methods in Applied Mechanics and Engineering*, 2005. Submitted.

[15] F. Shakib, T. J. R. Hughes, and Z. Johan. A new finite element formulation for computational fluid dynamics: X. The compressible Euler and Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, **89**:141–219, 1991.

[16] T. E. Tezduyar. Computation of moving boundaries and interfaces and stabilization parameters. *International Journal of Numerical Methods in Fluids*, **43**:555–575, 2003.